TRB MATHEMATICS

FUNTIONAL ANALYSIS

'Material Available with Question papers'

CLASS -I

Holder's inequality

If p> 1and $\frac{1}{p}+\frac{1}{q}=1$,then $\sum_{i=1}^{n}|x_i y_i| \leq [\sum_{i=1}^{n}|x_i|^p]^{\frac{1}{p}} [\sum_{i=1}^{n}|y_i|^q]^{\frac{1}{q}}$ OR

$\sum_{i=1}^{n}|x_i y_i| \leq [\sum_{i=1}^{n}|x_i| \quad] \quad [\sum_{i=1}^{n}|y_i| \quad]$

Holder's inequality For intgrable function

$\int_a^b |f(x)g(x)|$dx$\leq [\int_a^b |f(x)|^p dx]^{\frac{1}{p}} [\int_a^b |g(x)|^q dx]^{\frac{1}{q}}$

$\underline{put\ p = q = 2}$,then,

$\sum_{i=1}^{n}|x_i y_i| \leq [\sum_{i=1}^{n}|x_i|^2]^{\frac{1}{2}} [\sum_{i=1}^{n}|y_i|^2]^{\frac{1}{2}}$ or

$[\sum_{i=1}^{n}|x_i y_i|]^2 \leq [\sum_{i=1}^{n}|x_i|^2] \quad [\sum_{i=1}^{n}|y_i|^2]$

$This\ is\ known\ as\ cauchy'$s inequality

Minkowsk's inequality

- If p≥1,then $[\sum_{i=1}^{n}|x_i + y_i|^p]^{\frac{1}{p}} \leq [\sum_{i=1}^{n}|x_i|^p]^{\frac{1}{p}} + [\sum_{i=1}^{n}|y_i|^p]^{\frac{1}{p}}$

- If f and g are real or complex valued integrable function defined on [a,b], Then

$[\int_a^b |f(x)+g(x)|$ dx$]^p dx \leq [\int_a^b |f(x)|^p dx]^{\frac{1}{p}} + [\int_a^b |g(x)|^q dx]^{\frac{1}{q}}$ where p≥ 1

Metric space

Let X be a non-empty set. A metric on X is a real valued function X× X satisfying the following Three conditions,

For every x,y ∈ $X\ and\ x \neq y$

1. d(x,y)≥0 and d(x,y) = 0 if and only if x =y
2. d(x,y) = d(y,x) for every x,y∈ X
3. d(x, y) ≤d(x, z) + d(z,y) for any x,y,z ∈ X

   d(x,y) is called the distance between x and y ,it is finite non-negative real number.

Normed linear spaces

Let N be a complex or real linear space a norm on N is a function such that

(‖ ‖: $N \rightarrow R$)

i. $\|x\| \geq 0$ and $\|x\| = 0 \Leftrightarrow$ x = 0

ii. $\|x + y\| \leq \|x\| + \|y\|$

iii. $\|ax\| = |a|\|x\|$ for all x,yϵN and aϵc or R

N is called a normed linear space.

Definition

Let N be a normed linear space, a sequence $\{x_n\}$ in N is said to converge to an element x in N if given $\varepsilon > 0$,there exists a positive integer n_0 such that

$\|x_n - x\| < \varepsilon$ for all $n \geq n_0$

It is denoted by $\lim_{x\to\infty} x_n = x$

xn$\to x$ iff $\|x_n - x\| \to 0$ as $n \to \infty$

Theorems

- A normed linear space N is a matric space with respect to the metric d defined by

  D(x,y) = $\|x - y\|$ for all x,y ϵN

- If N is a normed linear space,Then
  - $|\|x\| + \|y\|| \leq \|x\| + \|y\|$
  - $|\|x\| - \|y\|| \leq \|x - y\|$

- If N is a normed linear space,Then the norm $\| \quad \|: N \to R$ is continuous on N.
- The operation of addition and scalar multiplication in N are jointly continuous.

  If $x_n \to x$, $y_n \to y$ and $a_n \to a$,Then $x_n + y_n \to x + y$, $a_n x_n \to ax$

- Let N be a normed linear space and M be a subspace of N, then the closure $\overline{M}$ of M is also a subspace of N
- A subset M in a normed linear space N is bounded if and only if there is a positive constant C such that $\|x\| \leq C$ for all x ϵM

Cauchy sequence

A sequence $\{x_n\}$ in N is called a Cauchy sequence in N ,If given $\varepsilon \geq 0$there exists a positive integer n_0 such that $\|x_n - x_m\| < \varepsilon$ for all $m,n \geq n_0$

If $\{x_n\}$ is a Cauchy sequence in N,Then $\|x_n - x_m\| \to 0$ as $m,n \to \infty$

Properties of a Cauchy sequence

i. If N is normed linear space ,then every convergent sequence is a Cauchy sequence.

It's converse is not true

ii. Every Cauchy sequence in a normed linear space is bounded.

Complete

A normed linear space N is said to be complete if every Cauchy sequence in N converges to an element of N.

$\Rightarrow$ If $\|x_n - x_m\| \to 0$ as,n$\to \infty$, then there exists x ϵN such that

$\|x_n - x\| \to 0$ as,n$\to \infty$,

Banach space

A complete normed linear space is called a Banach space.

- Every complete subspace M of a normed linear space is closed

Convergent of series

A series $\sum_{n=1}^{\infty} x_n$, x_n ϵN is said to be **convergent to x** ϵN, If the sequence of partial sums $\{s_n\}$ converges to x in N.

A series $\sum_{n=1}^{\infty} x_n$ is said to be **absolutely convergent** if $\sum_{n=1}^{\infty}\|x_n\|$ is convergent.

Theorem

A normed linear space N is complete if and only if every absolutely convergent series is convergent.

Example of Banach spaces.

1. The real linear space R and the complex linear space C are normed linear space under the norm $\|x\| = |x|$ for all xϵR $(or)C$

   R and C are complete $\Rightarrow$ R and C are Banach spaces.

2. The linear space R^n or C^n are Banach space with Norm, $\|x\| = [\sum_{i=1}^{n}|x_i|^2]^{\frac{1}{2}}$
3. (i) R^n or C^n are Banach space with Norm, $\|x\| = [\sum_{i=1}^{n}|x_i|^p]^{\frac{1}{p}}$,$1\leq p \leq \infty$
   Which is denited by l_p^n

   (ii) $\|x\|$ = Max $\{ |x_1|, |x_2|, |x_3|, \ldots ,|x_n|\}$, which is denoted by l_∞^n

4. The linear space C of all convergent sequence x = {x_n} with the Norm.

   $\|x\| = \sup_{1\leq n\leq\infty}|x_n|$ is a Banach space denoted by C

5. The linear space l_∞ of all bounded sequence x = {x_n} with the Norm.

   $\|x\| = \sup_{1\leq n\leq\infty}|x_n|$ is a Banach space.

6. The linear space l_p, p> 1 of all sequences $[\sum_{i=1}^{\infty}|x_i|^p] < \infty$ with norm

   $\|x\| = [\sum_{i=1}^{n}|x_i|^p]^{\frac{1}{p}}$ is a Banach space, It is denoted by $\| \quad \|_p$

7. If [a,b] is a bounded and closed interval, The linear space C[a,b] of all continuous functions defined on [a,b] is a Banach space with the norm,

   $\|f\| = \text{Sup}\{|f(x)| \,/\, x\epsilon[a.b]\}$

8. Let C(x) be the set of all continuous real valued function on a compact metric space X, then C(X) is a Banach space with the norm

   $\|f\| = \text{Sup}\{|f(x)| \,/\, x\epsilon X]$

Separable

A normed linear space N is said to be separable if it has a countable dense subset.

ie., There is a countable subset D in N such that $\overline{D} = N$

Example

1. Every subset of a separable normal linear space is separable
2. The normed linear space l_p, $1\leq p \leq \infty$ are separable
3. The space l_∞ is not separable

Quatient space

Let N be a normed linear space and M be a subspace of N, Then $\frac{N}{M} = \{x + M / x\epsilon \text{N}\}$ is called Quotient space.

It is denoted by Q(x)

Q(x) is called canonical(Natural) mapping of L onto $\frac{N}{M}$

Theorem:

If M is a closed linear subspace of a normed linear space N, Then quotian space $\frac{N}{M}$ is a normed linear space with norm of each cosert x+M defined as

$\|x+M\|$ = inf{ $\|x+m\|$ /mϵM }.

If N is Banach space, then the quotient space $\frac{N}{M}$ is also a Banach space with above norm

Direct sum of subspace

Let M and N are subspace of Banach space B, If every element z on B is represented uniquely in the form z = x+y ,xϵM,yϵN,Then B is said to be direct sum of N,M

It is denoted by B = M$\oplus$N

Theorem

Let a Banach Space B = M$\oplus$N and zϵB be z =x+y uniquely with xϵM,yϵN ,then

$\|z\|_1$ = $\|x\| + \|y\|$ is a normal on direct sum B = M$\oplus$N

If B1 is the direct sum space with this new norm, then B1 is a Banach space if M and N are closed.

Continuous linear Transformation

$T:N\rightarrow N^1$ is continuous if and only if $x_n \rightarrow x$ in N implies $T(x_n) \rightarrow T(x)$ in N^1

1. Zero Transformation is denoted by 0

2. Identity Transformation is denoted by I

Theorem

If T is continuous at the origin, Then it is continuous everywhere and the continutity is uniform.

Bounded linear transformation

A linear transformation $T:N\rightarrow N^1$ is said to be bounded linear transformation if there exists a positive constant M such that $\|T(x)\| \leq M\|x\|$ for all xϵN.

Theorem

1. $T:N\rightarrow N^1$ is bounded if and only if T is continuous.
2. Let $T:N\rightarrow N^1$ be a linear transformation ,Then T is bounded if and only if T maps bounded sets in N into bounded set in N^1

Bound of T

Let T be a bounded linear transformation of N into N^1, Then the norm ,

$\|T(x)\|$ = inf{M/ $\|T(x)\| \leq M\|x\|$ for all $x\epsilon N$} is called the bound of T (OR)

$\|T\| = \sup\{\frac{\|T(x)\|}{\|x\|} / x\epsilon N \text{ and } x \neq 0\}$

Theorem

If N and N^1 are normed linear space and $T:N\rightarrow N^1$, Then the following are equivalent

(a) $\|T\| = \sup\{\frac{\|T(x)\|}{\|x\|} / x\epsilon N \text{ and } x \neq 0\}$
(b) $\|T\| = \sup\{\|T(x)\| / x\epsilon N \text{ and } \|T\| \leq 1\}$
(c) $\|T\| = \sup\{\|T(x)\| / x\epsilon N \text{ and } \|T\| = 1\}$

B(N,N^1)

The set of all bounded linear transformation of normed space N into N^1 is denoted by $B(N,N^1)$

Theorem

- $B(N,N^1)$ is a normed linear space with linear operation
  (i) $(T_1+T_2)(x) = T_1(x)+T_2(x)$,

  (ii) $(aT)x = aT(x)$ and norm defined by $\|T\| = \sup\{\frac{\|T(x)\|}{\|x\|} / x\epsilon N \text{ and } x \neq 0\}$

  If N^1 is a Banach space ,then $B(N,N^1)$ is also Banach space.

1. If $T_1,T_2 \epsilon B(N,N^1)$, the

   $\|T_1T_2\| \leq \|T_1\|\|T_2\|$

2. If $T_n\rightarrow T$ and $T_n^1\rightarrow T^1$, Then $T_nT_n^1\rightarrow TT^1$ as $n\rightarrow \infty$ which implies that the multiplication is jointly continuous.

Theorem

- Let M be a closed subspace of a normed linear space and T be the natural mapping of N onto the quotient space $\frac{N}{M}$ defined by T(x) =x+M,Then T is bounded linear transformation with $\|T\| \leq 1$
- Let N and N^1 be normed linear space and let $T:N\rightarrow N^1$ be a bounded linear transformation of N into N^1,If M is the kernel of T, then
  i) M is closed subspace of N
  ii) T induces a natural transformation T^1 of N/M onto N^1 such that $\|T^1\| = \|T\|$

Definition

Let N and N^1 be normed linear space ,an **isometric isomorphism** of N into N^1 is a one-one linear transformation T of N into N^1 such that $\|T(x)\| = \|x\|$ for all $x \in N$

For any x,y $\in N \Rightarrow \|T(x) - T(y)\| = \|T(x-y)\| = \|x-y\|$

Definition

Topologically isomorphic

Two normed linear space N and N^1 are said to be topologically isomorphic, if

(i) There exists a linear operator $T:N \rightarrow N^1$ having the inverse T^{-1}
(ii) T establishes the isomorphism of N and N^1
(iii) T and T^{-1} are continuous in their respective domains.

Theorem

Let N and N^1 be normed linear space and Let T be linear transformation of N into $N^{1.}$ If T(N) is the range of T, Then the inverse T^{-1} exists and is bounded (continuous) in its domain of definition if and only if there exists a constan m>0 such that $\mathbf{m}\|\boldsymbol{x}\| \le \|\boldsymbol{T}(\boldsymbol{x})\|$ for all $x \in N$

Theorem

Let N and N^1 be normed linear space. The N and N^1 are topologically isomorphic if and only if there exist a linear operator T on N onto N^1 and positive constants m and M such that $\mathbf{m}\|\boldsymbol{x}\| \le \|\boldsymbol{T}(\boldsymbol{x})\| \le \mathbf{M}\|\boldsymbol{x}\|$ **for all** $x \in \boldsymbol{N}$

FUNCTIONAL ANALYSIS TEST – 1

1. If f and g are real or complex valued integrable function defined on [a,b], Then Minkowsk's inequality is

(a) $[\int_a^b |f(x)+g(x)|\,\mathrm{dx}]^p dx \le [\int_a^b |f(x)|^p dx]^{\frac{1}{p}} + [\int_a^b |g(x)|^q dx]^{\frac{1}{q}}$ where p< 1

(b) $[\int_a^b |f(x)+g(x)|\,\mathrm{dx}]^p dx \le [\int_a^b |f(x)| \quad dx] \quad + [\int_a^b |g(x)| \quad dx]$ wherep≥ 1

(c) $[\int_a^b |f(x)+g(x)|\,\mathrm{dx}]^p dx \le [\int_a^b |f(x)|^p dx]^{\frac{1}{p}} + [\int_a^b |g(x)|^p dx]^{\frac{1}{p}}$ where p≥ 1

(d) $[\int_a^b |f(x)+g(x)|\,\mathrm{dx}]^p dx > [\int_a^b |f(x)|^p dx]^{\frac{1}{p}} + [\int_a^b |g(x)|^q dx]^{\frac{1}{q}}$ where p≥ 1

2. If N be a complex or real linear space a norm on N is a function ,Then

(a) $\|x+y\| \leq \|x\|+\|y\|$ (b) $\|x+y\| > \|x\|+\|y\|$

(c) $\|x+y\|+\|y\|$ (d) $\|x+y\| \leq \|x\|$

3. Let N be a normed linear space, For every x,y $\in N$

(a) $|\|x\|-\|y\|| \leq \|x-y\|$ (b) $|\|x\|-\|y\|| > \|x\|-\|y\|$

(c) $\|x-y\|=|\|x\|-\|y\||$ (d) $|\|x\|-\|y\|| = 0$

4. If every Cauchy sequence in N converges to an element of a normed linear space N, then N is

(a) Banach space (b) complete (c) Hilbert space (d)Metric space

5. $l^n{}_p$ is

(a) Not Banach space (b) Linear space (c) Banach space (d) None of these

6. In a Banach space $x_n \rightarrow x$, $y_n \rightarrow y$ implies that $x_n + y_n \rightarrow$

(a) x+y (b) $\frac{x}{y}$ (c) x-y (d) xy

7. If N be a Normed linear space and $\|x\| = 0$ *if and only if*

(a) x= 0 (b) x is a real (c) x$\neq$ 0 (d) x>0

8. Every Cauchy sequence in a normed linear space is

(a) not converges (b) absolutely convergent

(c) bounded. (d)neither convergent nor divergent

9. A normed linear space N is complete if and only if every absolutely convergent series is,

(a) not converges (b) convergent

(c) divergent (d)neither convergent nor divergent

10. A subspace M of a Banach space B is complete if and only if M is

(a) bounded (b) Unbounded (c) Closed in B (d) Open in B

11. If M is a closed linear subspace of a normed linear space N, Then quotian

space $\frac{N}{M}$ is a normed linear space with norm

(a) $\|x+M\|$ = sup{ $\|x+m\|$ /mϵM } (b) $\|x+M\|$ = inf{ $\|x\|$ /xϵN }.

(c) $\|x+M\|$ = inf{ $\|x+m\|$ /mϵM } (d) $\|x+M\|$ = inf{ $\|m\|$ /mϵM }.

12. Let M be a closed subspace of a normed linear space N, For each xϵN,

let $\|x+M\|$ = inf{ $\|x+m\|$ /mϵM } then resfective to this norm

(a) N+M is a normed linear space (b)NM is a normed linear space

(c) $\frac{N}{M}$ is a normed linear space (d) N - M is a normed linear space

13. A complete normed linear space is

(a) Hilbert space (b) Banach space (c) Vector space (d) None of these

14. M is a closed linear subspace of the a normed linear space N. If N is a Banach space then the following is also a Banach space.

(a) NM (b) N+M (c) N-M (d) $\frac{N}{M}$

15. If p> 1 and q is defined by $\frac{1}{p}+\frac{1}{q}=1$ and for f and g two complex valued measurable function such that f∈ $L_p(x), g \in L_q(x)$, then the Holder's inequality is

(a) $\int_x |fg|$dx≤ $\|f\|_p \|g\|_q$ (b) $\left|\int_x fgdx\right| \leq \|f\|_p \|g\|_q$

(c) $\int_x |fg|$dx ≥ $\|f\|_p \|g\|_q$ (d) $\left|\int_x fgdx\right| \geq \|f\|_p \|g\|_q$

16. Let M be a subspace of a normed linear space N. The set of all cosets {x+M/ x∈ N } is a normed space in the quotient form if

(a) M is an open subspace of N (b) M = N

(c) M is a closed subspace of N (d) M is finite subspace os N

17. Let ($x_1, x_2, x_3, \ldots \ldots, x_n) \in R^n$. $\|x\| = (\sum_{i=1}^{n} |x_i|^p)^{\frac{1}{p}}$ does not define a norm when

(a) P = 100 (b) p = $\frac{3}{2}$ (c) p= 1 (d) p = $\frac{1}{2}$

18. Holder's inequality $\sum_{i=1}^{n} |x_i y_i| \leq [\sum_{i=1}^{n} |x_i|^p]^{\frac{1}{p}} [\sum_{i=1}^{n} |y_i|^q]^{\frac{1}{q}}$ for p,q such that ,

(a) p> 1and $p+q=1$ (b) p> 1and $\frac{1}{p}-\frac{1}{q}=1$

(c) p> 1and $\frac{1}{p}+\frac{1}{q}=0$ (d) p> 1and $\frac{1}{p}+\frac{1}{q}=1$

19. If $1 \leq P_1 < P_2 < \infty$,then

(a) $l_{p_1} \subset l_{p_2}$ and $\|x\|_{p_2} \geq \|x\|_{p_1}$ (b) $l_{p_1} \supset l_{p_2}$ and $\|x\|_{p_2} \leq \|x\|_{p_1}$

(c) $l_{p_1} \subset l_{p_2}$ and $\|x\|_{p_2} \leq \|x\|_{p_1}$ (d) $l_{p_1} \supset l_{p_2}$ and $\|x\|_{p_2} \geq \|x\|_{p_1}$

20. . The linear space l_∞ of all bounded sequence x = $\{x_n\}$ is Banach space with the Norm.

(a) $\|x\|$ = Max { $|x_1|, |x_2|, |x_3|, \ldots \quad, |x_n|$} (b) $\|x\| = \sup_{1 \leq n < \infty} |x_n|$

(c) $\|x\| = [\sum_{i=1}^{n} |x_i|^p]^{\frac{1}{p}}$ (d) None of these

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