TRB MATHEMATICS

ALGEBRA

CLASS -I

Equivalence relation

The binary relation $\sim$ on A is said to be an Equivalence relation on A if for all a,b,c in A

1.Reflex

$a \sim a$

2. Symmstry

$a \sim b \Rightarrow b \sim a$

3. Transity

$a \sim b \text{ and } b \sim c \Rightarrow a \sim c$

Examples:

1. Define $a \sim b$ for all a,b $\in S$ such that a = b , Than $\sim$ is Equivalence relation on S
2. Define $a \sim b$ for all a,b $\in S$ such that a – b is even integer , Than $\sim$ is Equivalence relation on S

Equivalence class

The Equivalence class of $a \in A$ is the set $\{ x \in A \setminus a \sim x \}$

It is denoted by cl(a)

Congruent modulo

Let n be a fixed positive integer. If a and b are integers such that a-b is divisible by n,We say that a is congruent to b modulo n and write $a \equiv b \pmod{n}$

Residue class modulo

$[a] = \{x \epsilon Z / \ x \equiv a \pmod{n} \}$

Mapping

If A and B are nonempty sets, than a mapping from A to B is a subset of $A \times B$ such that for every $a \in A$ there is aunique $b \in B$ such that $(a,b) \in A \times B$

map $f: A \rightarrow B$, f(a) = b, where b is unique in B

Onto mapping

The mapping $f: A \rightarrow B$ is said to be onto,if given $b \in B$ there exists an element $a \in A$ such that

$f(a) = b$

One-to-one mapping

The mapping $f:A \to B$ is said to be One-to-one mapping ,if whenever a=b,than f(a)=f(b) or $a \neq b$,than $f(a) \neq f(b)$

Composition(Product) of fuctions

If $f:A \to B$ *and* $g:B \to C$, than Composition of f and g is a map $g \circ f : A \to C$ defined by $(g \circ f)a = g[f(a)]$

Greatest common division (GCD)

The positive integer c is said to be Greatest common division of a and b if

(i). c is a division of a and b (c\a and c\b)

(ii). Any divisior of a and b is a divisor of c (d\a and d\b ⇒ d\c)

it is denoted by (a,b) = c

Relative prime

The integers a ,b are called relatively prime, if (a,b) = 1

- If a and b are non zero integers, than (a,b) exists and we can find integers m ,n such that (a,b) = ma+nb
- If a,b are relatively prime,than there exists m,n such that ma+nb = 1 ((a,b) = ma+nb=1)

Prime number

The integer $p > 1$ is a prime number If its only divisors are $\pm 1, \pm p$

- If a is relatively prime to b and a\bc, then a\c

Unique factorization

Any positive integer $a > 1$ can be factored in a unique way as $a = p_1^x p_2^y \dots p_n^z$ are prime numbers and each $x > 0$

Division Algorithm

Let a and b be integers, with b > 0. Then there exist unique integers q and r such that

$a = bq + r$ where $0 \leq r < b$.

GROUP:

A non-empty set G with abinary operation * is called agroup,If the following conditions are satisfied,

1.Closure: For all a,b ϵG ⇒a*b ϵG

2.Associative: For all a,b $, c$ ϵG ⇒a*(b*c) =(a*b)*c

3.Identity: For all a ϵG there exists an element e ϵG such that a*e =e*a =a

4.Inverse: For every a ϵG there exists an element a^{-1} ϵ G such that a*a^{-1} =a^{-1}*a =e

Abelian group (or) commutative group:

A group with commutative property is called an abelian group

That is, For all a,b ϵG ⇒a*b = b*a

Semi group : A set satisfying closure and associative which is called semi group.

Monoid: A set satisfying closure , associative ,identity which is called Monoid.

Oder of the Group

Total number of element in a Group is called order Group

Example:

1. (N ,+) ,(E,.) are semi group.
2. (N,.) ,(Z,.) ,(Q,.) ,(R,.) (C,.) are monoid.

3.(Z,+) ,(Q,+) ,(R,+) ,(C,+) ,(Q-{0} ,.) ,(R-{0},.) ,(C-{0} ,.)are abelian group.

4.The set of all unimodular complex numbers under multiplication of complex numbers is a group.

5.The set of all m×n matrices under the addition of matrices is an abelian group.

6. The set of all n×n non-singular matrices under the multiplication of matrices is finite abelian group.

7.4th root of unity {1,-1,i,-i} is an abelian group under multiplication.

8. {1,-1} is a Group under multiplication

8.3rd root or unity {1,ω,ω^2} is an abelian group under multiplication.

9.The set of all nth root of unitys under multiplication of complex number is an abelian group.

10. Z_5 = {[0],[1],[2],[3],[4]} ,the set of all residue of integer modulo 5 under addition modulo 5 is an abelian Group.

11.z_p is an abelian ,where p is prime number.

12.(E,+) is an abelian group, where E is the set of even numbers.

13.G ={2^n /nϵZ} is a group under multiplication.[identity 2^0 ,inverse of 2^n is 2^{-n}]

14.G ={f_1 ,f_2,f_3 ,f_4} defined by $f_1(z)$ =z,$f_2(z)$ =-z ,$f_3(z)$ =$\frac{1}{z}$,$f_4(z)$ =-$\frac{1}{z}$ is an abelian group under composition of mapping.[f1 is identity, inverse of f1 is f_1, inverse of f_2 is f_2, inverse of f_3 is f inverse of f4 is f4,]

15.(Z,*) is an finite abelian group where * is defined as a*b =a+b+2

Identity e =-2 ,Inverse of a is,a^{-1}=-a-4

16. set of all matrices of the form { $\begin{pmatrix} x & x \\ x & x \end{pmatrix}$ /xϵR-{0} } is a group under matrix multiplication.

Identity E = $\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$,Inverse of A = $\begin{pmatrix} x & x \\ x & x \end{pmatrix}$is,$A^{-1}$=$\begin{pmatrix} \frac{1}{4x} & \frac{1}{4x} \\ \frac{1}{4x} & \frac{1}{4x} \end{pmatrix}$,

17.G be the set of all rational number except 1 and * be defined on G by a *b =a+b-ab,then (G,*) is an abelian group. { Identity e =0 ,Inverse of a is,a^{-1}=$\frac{a}{a-1}$ }

18. Let G = S_3 be the set of one-one mappings of the set {x_1,x_2,x_3} onto itself ,It is a Group of order 6 under the product

19. Let n be a integer. G ={ a^i $/ i = 0,1,2,$ .... $(n-1)$} is a group under multiplication

20. Let G be the set of all 2× 2 matrices $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ where a,b,c,d are real numbers,such that ad-bc≠ 0 is a Group under multiplication.

21. Let G be the set of all 2× 2 matrices $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ where a,b,c,d are real numbers,such that ad-bc = 1 is a Group under multiplication.

22. Let G be the set of all 2× 2 matrices $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ where a,b,c,d are real numbers, not both zero,such that $a^2 + b^2 \neq 0$ is an abelian Group under multiplication.

23. Let G be the set of all 2× 2 matrices $\begin{bmatrix} a & b \\ 0 & d \end{bmatrix}$ where a,b,c,d are real numbers, not both zero,such that $ad \neq 0$ is an abelian Group under multiplication.

24. Let G be the set of all 2×2 matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ where a, b, c, d are integers modulo 2, such that ad - bc ≠0 Using matrix multiplications as the operation in G ,then G is a group of order 6.

Solution:

In the first row of any matrix belonging to G, each entrycould be 0 or 1 in Z_2, but (0, 0) should be extracted since ad - bc ≠ 0Hence we have 2^2 - 1 different choices for the first row. The second row is not a multiple of the first row. Hence G has (2^2 - 1)2 elements, namely 6.

$$G = \{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \}$$

25. Let G be the set of all 2×2 matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ where a, b, c, d are integers modulo 3, such that ad - bc ≠0 Using matrix multiplications as the operation in G ,then G is a group of order 48.

Solution:

In the first row of any matrix belonging to G, each entrycould be 0 or 1 in Z_2, but (0, 0) should be extracted since ad - bc ≠ 0Hence we have $3\times 3-1$ different choices for the first row. The second row is not a multiple of the first row. Second row (3×3)-3 possibilities Hence the number of elements in D is 8×6 = 48

Properties:

If G be the group,

1.The identity element of G is unique.

2.Every aϵG has aunique invers in G.

3.For every aϵG, $(a^{-1})^{-1}$=a.

4.For all a,b ϵG,$(a * b)^{-1}$ =b^{-1}* a^{-1}

5.For all a,b ϵG,

(i)a*b =a*c ⇒ b =c [left cancellation law]

(ii)b*a =c*c ⇒ b =c [right cancellation law]

6. For all a,b ϵG,the equation a*x =b and y*a =b have unique solution for x and y in G,the solutions are x = a^{-1*}b and y = b*a^{-1}.

7.$(a * b)^2$=$a^2 * b^2$ for all a,b ϵG iff G is an abelian group.

8.If every element of a group G is its own inverse, then G is an abelian .

9.every group of order FOUR is an abelian.

9. If G is an group in which $(a * b)^k = a^k * b^k$ for all three consecutive integers k and for all a,b ϵG,then G is an abelian .

10. If the Group G has three element ,it must be abelian.

11.A group having 4 or lessthan 4 elements is an abelian group.

12.If G is a finite group,then there exists a positive integer N such that a^N =e for all aϵG.

13. If G is a group of even order, prove that it has an element a ≠ e satisfying $a^2 = e$

14. If G is group of prime order,Than G is an abelian **(TRB-2004)**

SUBGROUP:

A non empty subset H of agroup G is called a subgroup of G if H itself form a group under the same operation defined on G.

Example

1.(E,+) is a subgroup of (Z,+)

2.{1,-1} is subgroup of {1,-1,i,-i}

3.(Z,+) is subgroup of (Q,+)

4. (Z,+) is subgroup of (R,+)

5. Zn is a subgroup of Z under addition,where n∈ Z

6. Let G be the Group of integers under addition,H the subset consisting of the multiples of 5 ,then H is a subgroup of G.

7. Let G be the Group of nonzero real numbers under multiplication, and let H be the sub set of positive rational numbers, then H is a subgroup of G.

8. Let a and b be integers.

Prove that the subset aZ + bZ = {ak + bl / l, k ∈ Z } is a subgroup of Z

9. Let G be the set of all 2×2 matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with ad - bc ≠0 Using matrix multiplication

10. Let H = {$\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \in G$ / ad ≠ 0 } ,then H is a subgroup of G

11. Let G be the Group of all nonzero complex numbers a+ib (a,b real, not both zero) under multiplication ,and H = { a+ib∈G / $a^2 + b^2 = 1$ } is a subgroup of G

The center of a Group

The center of a group G is Z(G) = {x ∈ G / ax = xa for all a ∈G}, Then Z(G) is a subgroup of G.

Normalizer (or) centralizer

N(a) = { x∈ $G\backslash ax = xa$ } is a subgroup of G and it is called Normalizer of G. N(a) is a subgroup of G

The center of G is the intersection of all the centralizer subgroups of G.

Theorem

- A non empty subset H of a group G is a subset of G ⇔ (i) a,b ϵH⇒a*b ϵH
  (ii)a ϵH ⇒ a^{-1} ϵH
- A non empty subset H of a group G is a subset of G ⇔ a,b ϵH⇒a*b^{-1} ϵ H
- If H is a non empty subset finite subset of a group G and H is closed under the product in G ,than H is a subgroup of G.
- If H and K are any two non empty subgroup of G,than $(H * K)^{-1}$ =$K^{-1} * H^{-1}$
- A non empty subset H of a group G is a subset of G,H is asubgroup of G iff HH =H and $H^{-1} = H$
- If H and K are subgroup of G, HK is subgroup of G iff HK =KH
- If H and K are subgroup of G,than H∩ K is also a subgroup of G
- Intersection of any number of subgroups of G is a subgroup of G
- H∪ K is a subgroup of G iff H⊂ K or $K \subset H$
- If H and K are subgroup of abelian group G, HK is subgroup of G.
- If H and K are two subgroup of a finite group G, and H⊆ K $Than [G: H] = [G: K][K: H]$
- If H and K are two finite subgroup of a group G and if O(H) ,O(K) are relatively prime,
  thanH∩ $K = \{e\}$
- If H and K are finite subgroup of G ,Than o(HK) = $\frac{o(H)o(k)}{o(H \cap K)}$
- If H and K are subgroup of a finite group G and o(H)> $\sqrt{o(G)}$,o(K))> $\sqrt{o(G)}$ than H∩ $K \neq \{e\}$
- aHa^{-1} = {aha^{-1}\h∈ H} is a sub group of G

QUESTIONS FOR FIRST CLASS WITH ANSWER

1. Which of the following is not a Group

   (a) (Z,+) (b) (Q,+) **(c) (R, .)** (d) (Q-{0} ,.)

2. (Z,*) is an finite abelian group where * is defined as a*b =a+b+2,then inverse element of a$\in G$ is,

   (a) a-4 **(b) –a-4** (c)-a+4 (d) –a-2

3. G = { $\begin{pmatrix} a & a \\ a & a \end{pmatrix}$: $a \in R - \{0\}$ } is a Group under multiplication,then inverse of $\begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix} is$

   (a) $\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$ (b) $\begin{pmatrix} \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} \end{pmatrix}$ **(c)** $\begin{pmatrix} \mathbf{\frac{1}{8}} & \mathbf{\frac{1}{8}} \\ \mathbf{\frac{1}{8}} & \mathbf{\frac{1}{8}} \end{pmatrix}$ (d) $\begin{pmatrix} -2 & -2 \\ -2 & -2 \end{pmatrix}$

4. Let G = $\{f_1, f_2, f_3, f_4\}$ is a Group under composition of the functions,then invers of $f_3$is,where$f_1(z) = z, f_2(z) = -z, f_3(z) = \frac{1}{z}, f_4(z) = -\frac{1}{z}$

   (a) f_1 (b) f_2 **(c) $\boldsymbol{f_3}$** (d) f_4

5. Let G be the set of all 2×2 matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ where a, b, c, d are integers modulo 3, such that ad - bc ≠0 Using matrix multiplications as the operation in G ,then G is a group of order

   (a) **48**. (b) 18 (c) 6 (d) 24

6. If G is a finite group of n, then for every a$\in G$, we have

   (a) $\boldsymbol{a^n = e}$ (b) $a^n = a^{-1}$ (c) $a^n = a$ (d) None of these

7. {1,-1} is a sub group of the group under multiplication

   (a) {1,I,-i} **(b) {1,-1,i,-i}** (c) {1,0,-1,i} (d) {-1,I,-I }

8. If e_1 and e_2 are two identity element of group G, then

   (a) $\boldsymbol{e_1}$ = $\boldsymbol{e_2}$ (b) $e_1 \neq e_2$ (c) e_1 = c e_2 (d)None of these

9. If G is a group ,then for all a,b $\in G$

   (a) $(ab)^{-1} = a^{-1}b^{-1}$ **(b) $\boldsymbol{(ab)^{-1} = b^{-1}a^{-1}}$** (c) $(ab)^{-1} = ab$ (d) $(ab)^{-1} = ba$

10. If G is a group , such that $(ab)^n = a^n b^n$ for three consecutive integers n for all a,b$\in G$,then G is

    (a) abelian (b) non-abelian (c) cyclic (d) additive group

11.Let G be the set of all 2×2 matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ where a, b, c, d are integers modulo 2, such that ad - bc ≠0 Using matrix multiplications as the operation in G ,then G is a group of order is,

(a) 2 (b) 3 (c) 4 **(d) 6**

12.If H_1 and H_2 are two subgroup of G, then folloeing is also a group of G

(a) $H_1 \cap H_2$ (b) $H_1 \cup H_2$ (c) H_1H_2 (d) None of these

13.If axa = b,then x is equal to

(a) $a^{-1}b$ (b) $a^{-1}b^{-1}$ (c) $a^{-1}b^{-1}b^{-1}$ **(d) $a^{-1}ba^{-1}$**

14.If G is a Group ,for a$\in G$,N(a) is the normalize of a, then for all x∈ N(a)

(a) xa=ax (b) xa = e (c) ax = e (d) xa≠ax

15.If G is a group such that $a^2 = e \; for \; all \; a \in G, then \; G \; is$

(a) abelian group (b)non abelian group (c) ring (d) field

16.If G is a group and a$\in G$,such that $a^2 = a$,then 'a' is equal to

(a) identity element (b) inverse (c) zero element (d) None of these

17.If H,K are two subgroup of G,then Hk is a subgroup of G ,iff

(a) HK =1 **(b) HK =KH** (c) $HK = H^{-1}K^{-1}$ (d) None of these

18.For all a,b ϵG,the equation a*x =b and y*a =b have unique solution for x and y in G,the solutions are

(a) x = a*b and y = ba (b) $x = ab^{-1}$ and $y = a^{-1*}b$

(c) $x = a^{-1*}b$ and $y = b*a^{-1}$. (d) $x = b*a^{-1}$ and $y = a^{-1*}b$

19.If H is a subgroup of G,then which of the following correct

(i) $H^{-1} = H$ (ii) $h \in H \Rightarrow h^{-1} \in H$ (iii) $H^{-1} \neq H$ (iv) $h^{-1} \in H^{-1}$ then $h \in H$

(a) (i),(ii) (b) (ii), (iii), (iv) **(c) (i),(ii), (iv)** (d) (i), (iv)

20.If H and K are two finite subgroup with order 6 and 5 of a group G ,then $O(H \cap K)$ is,

(a) **1** (b) 6 (c) 5 (d) 30